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NEWS OF THE ACADEMY OF SCIENCES OF THE USSR
(SELECTED ARTICLES)

FOREIGN TECHNOLOGY DIVISION

FOREIGN TECHNOLOGY DIVISION

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THERMOACOUSTICAL INSTABILITY OF A HETEROGENEOUS GAS FLOW

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K. I. Artamenov and I. G. Krutikova

The investigation of the stability of a one-dimensional single-velocity flow with distributed internal heat sources reduces to finding the characteristic values of the boundary-layer problem for a system of three ordinary differential equations.

The characteristic values have been numerically determined on digital electronic computers for two laws of heat conduction: 1) the heat supply rate is proportional to the density of the gas, and 2) the heat supply rate is proportional to the mass of the gas in a given cross section of the flow.

For a flow with a temperature gradient varying along its length (first case) the interaction of the pressure and entropy waves leads to the excitation of inherent acoustical oscillations in the flow; the fundamental note is the first to become unstable. The stability boundary is determined by the values of three dimensionless criteria: the ratio of the specific heats γ, the Mach number at the inlet M_0, and the criterion μ, characterizing the relationship between the heat supplied to the gas and the initial heat content of the gas. The last criterion determines both the steady-state temperature distribution

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along the length and the relationship between the heat supply rate and the density of the gas in the presence of oscillations.

A flow with constant temperature gradient along its length (second case) proves stable in the investigated intervals of the criteria.

Autoexcitation of oscillation in gas flows is a fairly frequent occurrence, an example of which could be the acoustical instability of the operational process in the combustion chambers of jet engines. The mechanism exciting the oscillations consists in the interdependence between the rate at which mass or energy is supplied to the flow and the velocity and thermodynamic parameters of the flow. This interdependence is usually realized in certain zones, the dimensions of which are small in comparison to the dimensions of the flow as a whole and, consequently, to the wave length of the oscillations.

In the schematization of the problem it is possible to introduce certain surfaces with well defined characteristics which separate the flow into regions for each of which the equations of homogeneous flow are applicable. When a one-dimensional flow is examined in this way, the pressure and entropy disturbances generated by the dividing surfaces are propagated independently of one another and may interact only at the boundaries of the homogeneous regions.

Under certain conditions these oscillations may increase through time. The problems of flow stability for this schematization have been examined at length in Rauschenbach's book [1].

In a number of cases this schematization is unsatisfactory and it is necessary to take into account the continuous variation of the parameters of the main flow. An example might be the problem of the stability of a one-dimensional flow with the heat source distributed along its entire length. For a description of the

oscillations in such a flow, it is necessary to make use of the complete energy equation, which expresses the continuous interdependence between the pressure waves propagating with the velocity of sound (acoustical waves) and entropy waves propagating at a velocity equal to the flow velocity. We note that such a flow may be considered concentrated when $\omega L/u \ll 1$, but not when $\omega L/c \ll 1$ as in the isoentropic case (u and c are the velocity of the flow and the velocity of sound respectively; ω is the oscillation frequency; and L is the characteristic length of the flow).

1. Equations of Heterogeneous Flow. Consider a one-dimensional flow of an ideal gas in a channel with varying cross-sectional area. Internal heat sources are present in the gas; the heat supply rate per unit length is denoted by Q.

When customary notations are used the flow equations have the following form:

$$\frac{1}{\rho}\frac{d\rho F}{dt} + \frac{\partial F u}{\partial x} = 0, \qquad \rho\frac{du}{dt} = -\frac{\partial P}{\partial x}, \qquad \frac{dP}{dt} - \gamma\frac{P}{\rho}\frac{d\rho}{dt} = \frac{\gamma - 1}{F}Q \tag{1.1}$$

The solution to the equations will be sought in the form

$$p = p^\circ(1+\eta), \qquad \rho = \rho^\circ(1+\delta), \qquad u = u^\circ(1+\nu), \qquad Q = Q^\circ(1+q) \tag{1.2}$$

The quantities with the superscript $^\circ$ characterizing the fundamental steady-state flow regime are found from Eq. (1.1), if we discard the first term on the right side in the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

The dimensionless small disturbances η, δ, ν, and q satisfy a system of three linear partial differential equations:

$$\frac{\partial\delta}{\partial x} + u^\circ\left(\frac{\partial\nu}{\partial x} + \frac{\partial\delta}{\partial x}\right) = 0, \qquad \frac{\partial\nu}{\partial t} + u^\circ\frac{\partial\nu}{\partial x} + \frac{p^\circ}{\rho^\circ u^{\circ}}\frac{\partial\eta}{\partial x} + \frac{du^\circ}{dx}(2\nu + \delta - \eta) = 0$$

$$\frac{\partial\eta}{\partial t} - \gamma\frac{\partial\delta}{\partial t} + u^\circ\left(\frac{\partial\eta}{\partial x} - \gamma\frac{\partial\delta}{\partial x}\right) = \Omega(q - \eta - \nu) \qquad \left(\Omega = \frac{Q^\circ}{\rho^\circ F c_p T^\circ}\left[\frac{1}{\sec}\right]\right) \tag{1.3}$$

Here Ω is the characteristic frequency of the heat supply process. When $\Omega = 0$ the entropy equation is solved separately from the other two; if $\Omega \neq 0$, then this quantity determines the degree to which the entropy oscillations are interdependent upon the other flow parameters.

We note that for isoentropic distributed mass and momentum transfer the flow would be described by a system of second order differential equations, instead of third order as in our case.

We shall limit our consideration to a flow with small acceleration; i.e., we shall set $du^*/dz = 0$. In this case the area of the channel cross section is determined by the heat supply law. Let us consider the two heat transfer laws:

the heat supply rate is proportional to the density of the gas

$$Q = k_1\rho \quad \text{(variant 1)} \tag{1.4}$$

the heat supply rate is proportional to the mass of gas in a given cross section

$$Q = k_2\rho F \quad \text{(variant 2)} \tag{1.5}$$

We shall assume that the static heat supply law is preserved in the presence of oscillations; then we will obtain for both cases:

$$q = \delta \tag{1.6}$$

The coefficients in the equations are not explicit functions of time, and therefore the solution may be sought in the form

$$\eta = \eta(z)e^{\omega t}, \qquad \delta = \delta(z)e^{\omega t}$$

Let us introduce the dimensionless coordinate $\xi = z/L$ and the dimensionless time $\tau = tu_0/L$; then Eqs. (1.3) may be reduced to canonical form

$$
\begin{aligned}
(\gamma - M^2)\frac{d\delta}{d\xi} &= \delta(-r-\gamma\lambda) + \nu(-M^2 r + \gamma\lambda) + \eta(\gamma M^2 \lambda r + \gamma\lambda)\\
(\gamma - M^2)\frac{d\nu}{d\xi} &= \delta\gamma\lambda + \nu(M^2 r - \gamma\lambda) + \eta(-r-\gamma\lambda)\\
(\gamma - M^2)\frac{d\eta}{d\xi} &= -\delta\gamma M^2\lambda + \nu(-\gamma M^2 r + \gamma M^2\lambda) + \eta(M^2 r + \gamma\lambda M^2)
\end{aligned}
\tag{1.7}
$$

Here

$$
M^2 = \frac{u_0^2}{p^\circ/\rho^\circ}, \qquad \lambda = \Omega\frac{L}{u_0} = \frac{Q^\circ L}{\rho^\circ F u_0 C_p T^\circ}, \qquad r = \frac{\omega L}{u_0}
$$

The quantity M in our case is the isothermic Mach number.

The coefficients M^2 and λ are determined by the heat supply law. Making use of the equation of state of an ideal gas we obtain from the steady-state equations

$$
M^2 = \frac{M_0^2}{\sqrt{1+\mu\xi}}, \qquad \lambda = \frac{\mu}{2(1+\mu\xi)}, \qquad \mu = \frac{2Q_0 L}{\rho_0 u_0 F_0 C_p T_0}
\tag{1.8}
$$

for variant 1, and

$$
M^2 = \frac{M_0^2}{(1+\mu\xi)}, \qquad \lambda = \frac{\mu}{1+\mu\xi}, \qquad \mu = \frac{Q_0 L}{\rho_0 u_0 F_0 C_p T_0}
\tag{1.9}
$$

for variant 2.

We use the subscript 0 to denote the values at the inlet (when $\xi = 0$). For the boundary conditions we take the following:

$$
\begin{aligned}
&\nu + \delta = 0, \quad \eta - \gamma\delta = 0 \qquad \text{when } \xi = 0\\
&\nu + {}^1\!/_2\delta - {}^1\!/_2\eta = 0 \qquad \text{when } \xi = 1
\end{aligned}
\tag{1.10}
$$

The first and second conditions express the constancy of flow rate and entropy respectively at the inlet to the channel. The third condition expresses the constancy of the Mach number at the outlet and corresponds to quasisteady discharge through a critical nozzle.

The boundary condition at the inlet into a supersonic nozzle was given more precisely by Crocco [2]; we shall confine ourselves to the simple relationships listed above, which are valid when the length of the convergent section of the nozzle is small in comparison to the

over-all length of the channel.

2. Method of Solution. The problem of investigating the flow stability reduces itself to a determination of the conditions for which the real part of the characteristic values of the boundary-layer problem (1.7)-(1.10) pass through zero. When Re $r > 0$ the flow is unstable; when Re $r < 0$ it is stable.

The characteristic values of the problem in some domain D of the complex variable $r = x + iy$ are found in the following way. The domain D is covered with a network of values of x and y. For each of these values Cauchy's problem is solved for system (1.7) with the initial conditions for $\xi = 0$.

$$\nu + \delta = 0, \qquad \eta - \gamma\delta = 0, \qquad \delta = 1 \tag{2.1}$$

Naturally, with these initial conditions the third boundary condition in (1.10) was not fulfilled; when $\xi = 1$ we obtain:

$$\nu + {}^1/_2\delta - {}^1/_2\eta = \zeta_1(x, y) + \zeta_2(x, y) \tag{2.2}$$

The characteristic numbers of the boundary-value problem under consideration are those values of $r = x + iy$ for which simultaneously

$$\zeta_1(x, y) = 0, \qquad \zeta_2(x, y) = 0 \tag{2.3}$$

The calculation of Cauchy's problem was programmed on a Strela electronic computer (subsequently other machines were used also).

The characteristic values were determined graphically for chosen values of γ, M_0, and μ using the values of $\zeta_1(x,y)$ and $\zeta_2(x,y)$ obtained on the machine. The accuracy of the characteristic values was determined by the accuracy of the solution to the Cauchy problem. The solution to system (1.7) is fairly complicated in form. The function

$\delta(\xi)$ is a rapidly oscillating function since a term containing $e^{-r\xi}$ is included in its solution (the period of the oscillations is $2\pi/y$).

The function $\nu(\xi)$ also varies rapidly, but its shape is smoother than that of $\delta(\xi)$ since the coefficient of $\nu(\xi)$ in the second equation of (1.7) is M_0^2 times less than that of $\delta(\xi)$ in the first equation. The function $\eta(\xi)$ varies little by comparison to $\delta(\xi)$ and $\nu(\xi)$, since the coefficients in the equation for $\eta(\xi)$ are approximately M_0^2 times less than the coefficients in the equations for $\delta(\xi)$ and $\nu(\xi)$. In addition to these, a number of the coefficients in the equations are far from smooth (the sharp maximum when $\xi = 0$ for λ when μ is large, etc.).

The equations were integrated over each division interval of the ξ axis in the following way.

The function $\delta(\xi)$ was determined by exact integration of the first equation in (2.7) with $\eta(\xi)$ replaced by a constant and $\nu(\xi)$ replaced by a linear function. The function $\nu(\xi)$ was determined from the second equation by exact integration after substitution of $\delta(\xi)$ found previously. The equation for $\eta(\xi)$ was computed using Fuler's method on an average layer.

The method taken for the calculation possesses slow convergence for large y, and therefore for acceptable accuracy the number of division intervals N for the majority of variants was chosen in accordance with the formula

$$N = 40 + [y/25]$$

3. <u>Characteristic Values and the Stability Boundary</u>. In some cases exact solutions to the boundary-value problem have been obtained. For the case $\mu = 0$;

$$x = \frac{1 - M_x^2}{2M_x} \ln \left[\frac{(1 - M_x)(1 + \chi M_x)}{(1 + M_x)(1 - \chi M_x)} \right]$$

$$\left(M_x = \frac{1}{\sqrt{\gamma}} M_0, \ \chi = \frac{\gamma - 1}{2\gamma} \right) \tag{3.1}$$

$$y = k\pi \frac{1 - M_x^2}{M_x} \quad (k = 0, 1, \ldots) \tag{3.2}$$

When $M^2 \ll 1$ the frequency of the oscillations corresponds to the characteristic acoustical frequencies of the tube:

$$\alpha = M_x y = \frac{\omega L}{\sqrt{\gamma} C_0} = k\pi \tag{3.3}$$

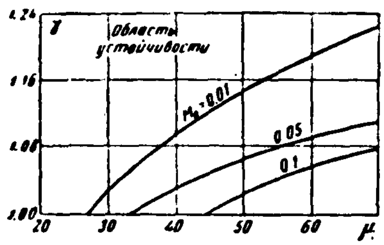

Fig. 1.

For $\mu = 0$ but when the second condition is changed to $\eta - \gamma_1 \delta = 0$, $\gamma_1 \neq \gamma$ for $\xi = 0$, there occur in addition to the "fundamental" values of (3.2), a large number of characteristic numbers for which $y \approx 2n\pi (n = 1, 2 \ldots)$.

There is also an analogous exact solution for small μ when the coefficients of Eqs. (2.7) are replaced by certain average values. Obviously, the new characteristic values having periods with respect to y equal to $2n\pi$ correspond to the entropy waves. As evaluations have shown, the damping coefficients for them for the values of μ that have been studied lie well into the negative region $(x \leqslant -3)$, therefore these characteristic values were not taken into account when finding the stability regions.

It follows from Eqs. (1.7)-(1.10) that, when the law for the steady-state temperature distribution along the length is given, the stability of the flow is determined by three dimensionless criteria: 1) γ — the specific heat ratio; 2) M_0 — the inlet Mach number of the flow; 3) μ — the ratio (doubled for variant 2) of the instantaneous heat transfer over the entire length of the channel for constant Q_0 (equal to the heat transfer rate in the inlet cross section) to the initial heat content of the instantaneous rate of flow of the gas. The parameter μ may be expressed in terms of the ratio of the initial and final temperatures:

$$\mu^{(1)} = \left(\frac{T_k}{T_0}\right)^2 - 1, \qquad \mu^{(2)} = \left(\frac{T_k}{T_0}\right) - 1 \tag{3.4}$$

We note that μ characterizes both the steady-state flow parameters and the heat supply rate in the presence of oscillations for the adopted quasi-steady dependence of Q on ρ. The velocity u_0 enters into both μ and M_0. Obviously, when $u_0 \to 0 (\mu \to \infty, M_0 \to 0)$ the steady-state regime is not possible.

The machine computation was carried out for several values of γ $(1 \leqslant \gamma \leqslant 1.4)$, M_0 $(0.01 \leqslant M_0 \leqslant 0.1)$, and for values of μ from zero to 70 and above. The fundamental characteristic values of the problem are infinitely numerous, the calculation was performed for the first three (k = 1, 2, and 3 in (3.2)).

The computations showed than when μ is increased the characteristic values move in a certain way in the xy plane, approaching the y-axis and tending toward some limiting value as $\mu \to \infty$.

The least damping during the increase in μ is observed in the fundamental note of the oscillations (k = 1).

For variant 1 (variable temperature gradient) stability boundaries in the $\mu\gamma$ plane for several values of M_0 have been presented in Fig. 1.

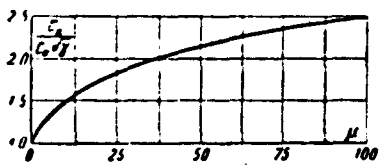

Fig. 2.

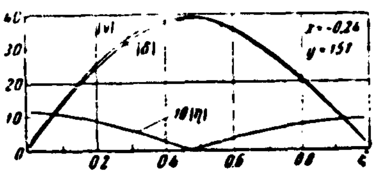

Fig. 3.

(Values of γ close to unity are possible for flows with dissociation or ionization). The frequency of the oscillations ω for all M_0 and γ on the stability boundary are described with high accuracy by Eq. (3.2) when $\sqrt{\gamma}C_0$ is replaced by the average sound velocity C_x:

$$C_x = \int_0^1 \sqrt{\gamma} C(\xi)\, d\xi = \sqrt{\gamma} C_0 \frac{(1+\mu)^{5/4} - 1}{5\mu/4} \tag{3.5}$$

A curve of $C_x/\sqrt{\gamma}C_0$ is given in Fig. 2.

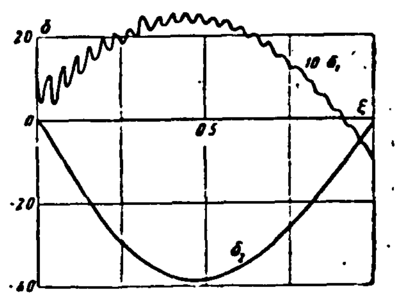

Fig. 4.

The possibility of introducing the average sound velocity indicates a coincidence between the velocity of the propagation of the oscillations and the velocity of sound for the given temperature distribution. The velocity of the propagation may be different when other distribution laws apply [3].

As follows from Fig. 1, the flow is stabilized during the increase

in M_0 and γ and the decrease in μ. Presented in Fig. 3 are the averaged absolute values of the characteristic functions which lie close to the distribution of the amplitudes of the neutral oscillations along the length of the channel (when $M_0 = 0.1$, $\gamma = 1.2$, and $\delta(0) = 1$). In contrast to isoentropic oscillations, for which $\eta = \gamma\delta$, the amplitudes of the density (and velocity) oscillations greatly exceed the amplitudes of the pressure oscillations. Presented in Fig. 4 are the calculated real and imaginary parts of the density oscillations $\delta = \delta_1 + i\delta_2$. Present in the curve for δ_1 are the oscillations which are superimposed over the fundamental oscillation. These oscillations have a shorter wavelength (by about M_0 times) than that of the fundamental oscillation and correspond to isobaric entropy disturbances (these oscillations are not noticeable in the curves of η_1 and η_2).

For variant 2 (constant temperature gradient along the length) no unstable oscillations were detected even for the larger values of μ (up to 200). This attests to the strong influence on flow stability of the steady-state heat supply law. The flow is also stable for $q = 0$, when small disturbances do not alter the heat supply rate.

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NON-STEADY FLUTTER OF PLATES AND SLANTED SHELLS IN GAS SHELLS

V. V. Bolotin

The problem of the unsteady flutter of plates and slanted shells in supersonic gas streams is examined. The examination is based on equations which were obtained in previous papers [1 and 2] and later applied to a number of steady problems regarding oscillations and thermal buckling [3-7]. The present article deals with finding non-steady solutions for the simultaneous variation of stream velocity, ambient medium density, plate or shell temperature, etc. For this an approximate method for investigation of non-steady processes in linear systems with many degrees of freedom [8], hereafter referred to as the "method of single-frequency oscillations" is employed. An example is given of a determination of the amplitudes of non-steady flutter for a flat panel.

1. Initial Equations. By the method described in previous papers [1, 2, and 6] the equations for the oscillations of plates and shells lying in a temperature field and flowed past by a supersonic gas stream may be reduced to the following system of equations:

$$\ddot{\zeta}_j + g_j\dot{\zeta}_j + \omega_j^2\zeta_j + 0\sum_{k=1}^{n} a_{jk}\zeta_k + \mu\sum_{k=1}^{n} b_{jk}\zeta_k = \tag{1.1}$$
$$= \Phi_j(\zeta_1, \zeta_2, \ldots, \zeta_n, \dot{\zeta}_1, \dot{\zeta}_2, \ldots, \dot{\zeta}_n, \mu, 0) \quad (j = 1, 2, \ldots, n)$$

Here ζ_j are generalized coordinates (the dimensionless coefficients in the expansion of the additional normal deflection of the plate or slanted shell into series with respect to the n first modes of the natural oscillations in a vacuum; g_j and ω_j are the dimensionless damping characteristics and the dimensionless frequencies of the respective oscillation modes; a_{jk} and b_{jk} are certain coefficients depending on the nature of the temperature field and the properties of the flow; Φ_j; are certain linear functions, which are polynomials of the generalized coordinates and generalized velocities. The dots in Eqs. (1.1) signify differentiation with respect to dimensionless time t. The characteristic panel temperature and unperturbed stream velocity are described by the parameters θ and μ respectively. We shall choose these parameters in the following way [3 and 6]:

$$\theta = \frac{12}{\pi^2}(1+\nu)\alpha T\left(\frac{a}{h}\right)^2, \qquad \mu = \frac{48\varkappa}{\pi^4}(1-\nu^2)\frac{p_1}{E}\left(\frac{a}{h}\right)^3 M \tag{1.2}$$

where T is the characteristic temperature; α is the thermal expansion coefficient of the panel material; ν is the Poisson ratio; E is the elasticity modulus; a is the characteristic dimension of the middle surface; h is the thickness of the panel; $\varkappa$ is the polytrope index of the flow; p_1 is the pressure of the unperturbed flow; and M is the Mach number for the unperturbed flow (by unperturbed flow we mean flow in the absence of flutter or thermal buckling of the paneling). It is possible to show that the parameters θ and μ quite completely describe the conditions of the behavior of paneling in a variable temperature, velocity, and pressure field. The values of θ and μ at any moment of time yield a certain image point on the θ and μ plane.

The linear system corresponding to (1.1)

$$\ddot{\zeta}_j + g_j\dot{\zeta}_j + \omega_j^2\zeta_j + \theta\sum_{k=1}^{n} a_{jk}\zeta_k + \mu\sum_{k=1}^{n} b_{jk}\zeta_k = 0 \quad (j=1,2,\ldots,n) \tag{1.3}$$

has, apart from the trivial solution $\zeta_1 = \zeta_2 = \ldots = \zeta_n = 0$, a solution of the type

$$\zeta_j = \xi_j e^{\sigma t} \tag{1.4}$$

where ξ_j and σ are certain constants. Substituting (1.4) into Eq. (1.3) we obtain the characteristic equation

$$|(\sigma^2 + g_j\sigma + \omega_j^2)\,\delta_{jk} + \theta a_{jk} + \mu b_{jk}| = 0 \tag{1.5}$$

(δ_{jk} is the Kronecker symbol). The trivial solution is stable as long as all Re $\sigma < 0$; it becomes unstable if just one of the roots of Eq. (1.5) passes over into the right half-plane.

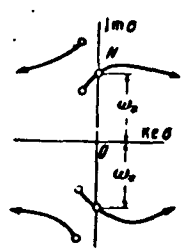

Fig. 1.

Hereafter it will be necessary to distinguish between two types of transitions to instability. First, the transition may take place through a point on the imaginary axis not coincident with the origin (Fig. 1). In this case Eq. (1.5) has on the stability boundary two purely imaginary roots $\sigma = \pm i\omega_*$, where ω_* is the flutter frequency. On the $\mu\theta$ plane the set of points corresponding to this type of instability form a line $f(\mu_*, \theta_*) = 0$, which we shall call the line of dynamic instability or the flutter line. Second, a transition through the origin is possible (Fig. 2). In this case Eq. (1.5) has a root $\sigma = 0$, and the stability loss is of a non-oscillatory nature.

The line $s(\mu_*, \theta_*) = 0$ on which one of the roots of Eq. (1.5) has the root $\sigma = 0$, while the remaining roots lie in the left half-plane will be called the <u>line of static instability</u>.

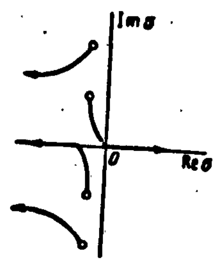

Fig. 2.

The arrangement of these lines for the case of a flat rectangular plante [3] is shown in Fig. 3.

When the transition takes place through the line AF, flutter ensues, and when it takes place through the line DF, a thermal buckling of the plate occurs. Corresponding to a variation in flight conditions are fairly complex trajectories, which the image point describes on the $\mu\theta$ plane.

2. <u>Use of the Method of Single-Frequency Solutions for Analysis of the Behavior of a System Close to the Flutter Line</u>. Let us assume that the image point intercepts the flutter line $f(\mu_*, \theta_*) = 0$ at some point N (Fig. 3). Let ω_* be the frequency of the oscillations at the flutter boundary. If the boundary is crossed slowly enough and if the image point does not go too far from the boundary, then it is to be expected that the oscillations of the system will be close to periodic oscillations with frequency ω_*.

We shall use this idea as the basis for an approximate solution of system (1.1) for variable μ and θ.

After writing system (1.1) in the form

$$\ddot{\zeta}_j + g_j\dot{\zeta}_j + \omega_j^2\zeta_j + \theta(\tau)\sum_{k=1}^{n} a_{jk}\zeta_k + \mu(\tau)\sum_{k=1}^{n} b_{jk}\zeta_k = \\ = \Phi_j[\zeta_1, \zeta_2, \ldots \zeta_n, \dot{\zeta}_1, \dot{\zeta}_2, \ldots \dot{\zeta}_n, \mu(\tau), \theta(\tau)] \quad (j = 1, 2, \ldots, n) \tag{2.1}$$

we shall assume that the parameters θ and μ are functions of the "slow" time $\tau = \varepsilon t$ $(\varepsilon \ll 1)$. Furthermore, we shall assume that the "mistunings" $\mu - \mu_*$ and $\theta - \theta_*$ are sufficiently small. For the time being we shall make no assumption concerning the smallness of the damping characteristics g_j; as we shall see below, this is because of certain peculiar features of the problem under consideration. As before, the dots in Eqs. (2.1) signify differentiation with respect to "fast" time $\underline{t}$.

Let us transform Eq. (2.1) in the following way:

$$\ddot{\zeta}_j + g_j\dot{\zeta}_j + \omega_j^2\zeta_j + \theta_*\sum_{k=1}^{n} a_{jk}\zeta_k + \mu_*\sum_{k=1}^{n} b_{jk}\zeta_k = \\ = \varepsilon\Psi_j[\zeta_1, \zeta_2, \ldots, \zeta_n, \dot{\zeta}_1, \dot{\zeta}_2, \ldots, \dot{\zeta}_n, \mu(\tau), \theta(\tau)] \quad (j = 1, 2, \ldots, n) \tag{2.2}$$

where

$$\varepsilon\Psi_j = \Phi_j[\zeta_1, \zeta_2, \ldots, \zeta_n, \dot{\zeta}_1, \dot{\zeta}_2, \ldots, \dot{\zeta}_n, \mu(\tau), \theta(\tau)] - \\ - (\theta - \theta_*)\sum_{k=1}^{n} a_{jk}\zeta_k - (\mu - \mu_*)\sum_{k=1}^{n} b_{jk}\zeta_k \tag{2.3}$$

We introduce here the small parameter ε in accordance with the above made assumptions. It may be assumed that the point N "wanders" along the flutter line: $\theta_* = \theta_*(\tau)$, $\mu_* = \mu_*(\tau)$. For example, it is advantageous to choose it so that the overall mistuning $[(\mu - \mu_*)^2 + (\theta - \theta_*)^2]^{1/2}$ be minimized (Fig. 4), etc. But here for simplicity we shall assume that $\mu_* = \text{const}$ and $\theta_* = \text{const}$.

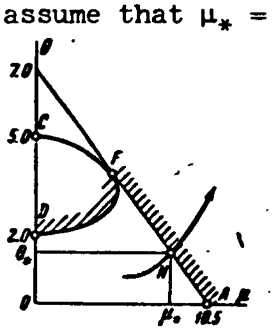

Fig. 3.

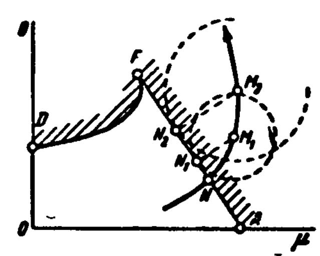

Fig. 4.

Let us consider the generating system corresponding to (2.2)

$$\ddot{\zeta}_j + g_j\dot{\zeta}_j + \omega_j^2\zeta_j + \theta_* \sum_{k=1}^{n} a_{jk}\zeta_k + \mu_* \sum_{k=1}^{n} b_{jk}\zeta_k = 0 \quad (j = 1, 2, \ldots, n) \tag{2.4}$$

We note the following characteristics of system (2.4):

1) system (2.4) admits of two linearly independent periodic solutions with frequency ω_*. In fact, Eq. (1.5) has two purely imaginary roots $\sigma = \pm\, \omega_*$ on the flutter boundary. Consequently, it is possible to construct a solution to system (2.4) as a function of two real constants.

2) The only steady-state solution to system (2.4) is the solution $\zeta_1 = \zeta_2 = \ldots = \zeta_n = 0$.

3) The frequency ω_* is not multiple, i.e., at the point N (Fig. 1) only one of the characteristic indices emerges onto the right half-plane. We note that if the system

$$\ddot{\zeta}_j + \omega_j^2\zeta_j + \theta_* \sum_{k=1}^{n} a_{jk}\zeta_k + \mu_* \sum_{k=1}^{n} b_{jk}\zeta_k = 0$$

were taken as the generating system (which is natural in view of the relative slightness of the damping), this condition would not be fulfilled. This peculiarity, which is illustrated in Fig. 5, was first noted in one of the author's previous papers [2]. The condition of the absence of multiple frequencies is a portion of the stronger condition of the absence of "inherent resonances". But if the degree of instability is equal to two; i.e., if all the characteristic indices except two lie in the left half-plane, the condition that has been formulated is sufficient.

It is easy to see that the theory of "single-frequency oscillations" in systems with many degrees of freedom developed by N. M. Krylov, N. N. Bogolyubov, and Yu. A. Mitropol'skiy [8] is entirely applicable to system (2.2). The generating solutions are of the form

$$\zeta_j^{(1)} = \xi_j e^{i\omega_* t}, \qquad \zeta_j^{(2)} = \xi_j^* e^{-i\omega_* t} \tag{2.5}$$

(the superscript asterisk denotes conversion to the complex conjugate). The complex constants ξ_j are found as a non-trivial solution to the algebraic equations

$$(\omega_j^2 + ig_j\omega_* - \omega_*^2)\xi_j + \theta_* \sum_{k=1}^{n} a_{jk}\xi_k + \mu_* \sum_{k=1}^{n} b_{jk}\xi_k = 0 \qquad (j = 1, 2, \ldots, n) \tag{2.6}$$

We shall seek a solution to the non-linear system (2.2) close to the solution to (2.5) in the form

$$\zeta_j = a\xi_j e^{i\psi} + a\xi_j^* e^{-i\psi} + \varepsilon u_j^{(1)}(a, \psi) + \varepsilon^2 u_j^{(2)}(a, \psi) + \ldots \tag{2.7}$$

Here a and $\varphi = \psi - \omega_* t$ are the slowly varying amplitude and phase (real functions of the "slow" time τ), and $u_j^{(1)}$, $u_j^{(2)}$,... are the amplitudes of a. The functions $a(\tau)$ and $\psi(\tau)$ are determined from the equations

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots$$
$$\frac{d\psi}{dt} = \omega_* + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots \tag{2.8}$$

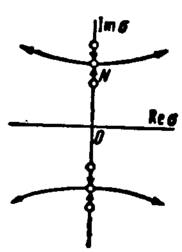

Fig. 5.

It is necessary to set up an algorithm for finding $u_j^{(1)}$, $u_j^{(2)}$,...,A_1, A_2,..., B_1, B_2, ..., such that the expressions (2.7) satisfy Eqs. (2.2) whenever a and ψ satisfy Eqs. (2.8). Hereafter we shall confine ourselves to the first approximation The equations for the first approximation can

be derived from (2.2) by a somewhat simpler, though more formal, method than was done in Bogolyubov's paper [8]. In the present article we shall choose this simpler method.

3. Derivation of the Equations for Amplitude and Phase in Non-Steady Flutter. In contrast to (2.7) the solution to (2.2) will be sought in the form

$$\zeta_j = a\eta_{j1}e^{i\psi} + a\eta_{j2}e^{-i\psi} \tag{3.1}$$

where a and ψ are real functions, and

$$\begin{aligned}\eta_{j1} &= \xi_j + \varepsilon v_{j1}^{(1)}(a, \psi) + \varepsilon^2 v_{j1}^{(2)}(a, \psi) + \ldots \\ \eta_{j2} &= \xi_j^* + \varepsilon v_{j2}^{(1)}(a, \psi) + \varepsilon^2 v_{j2}^{(2)}(a, \psi) + \ldots\end{aligned} \tag{3.2}$$

If the desired solution is close to a harmonic with frequency ω_*, then expressions (3.2) may obviously be treated as functions of "slow" time τ. Taking this into account we may write down the following approximate relationships:

$$\begin{aligned}\dot{\zeta}_j &= [\dot{a} + ia(\omega_* + \dot{\varphi})]\eta_{j1}e^{i\psi} + [\dot{a} - ia(\omega_* + \dot{\varphi})]\eta_{j2}e^{-i\psi} + \ldots \\ \ddot{\zeta}_j &= [2i\dot{a}\omega_* - a(\omega_*^2 + 2\omega_*\dot{\varphi})]\eta_{j1}e^{i\psi} + \\ &\quad + [-2i\dot{a}\omega_* - a(\omega_*^2 + 2\omega_*\dot{\varphi})]\eta_{j2}e^{-i\psi} + \ldots\end{aligned} \tag{3.3}$$

The dots signify terms containing ε^2, ε^3, etc. Expressions (3.3) are substituted into the left side of Eqs. (2.2). The expressions

$$\zeta_j = a\xi_je^{i\psi} + a\xi_j^*e^{-i\psi} + \ldots \qquad \dot{\zeta}_j = ia\omega_*\xi_je^{i\psi} - ia\omega_*\xi_j^*e^{-i\psi} + \ldots \tag{3.4}$$

are substituted into the right side, since it contains a factor ε.

Expanding the right side into a Fourier series

$$\Psi_j = \Psi_j^{(1)}e^{i\psi} + \Psi_j^{(2)}e^{-i\psi} + \ldots$$

are equating coefficients exp $i\psi$, we obtain the system of equations

$$\begin{aligned}(\omega_j^2 - \omega_*^2)a\eta_{j1} + i\omega_*g_ja\eta_{j1} + \theta_*a\sum_{k=1}^{n} a_{jk}\eta_{k1} + \mu_*a\sum_{k=1}^{n} b_{jk}\eta_{k1} = \\ = -2i\omega_*\dot{a}\xi_j + 2\omega_*\dot{\varphi}a\xi_j - g_j\dot{a}\xi_j - ig_j\dot{\varphi}a\xi_j + \Psi_j^{(1)}\end{aligned} \tag{3.5}$$

Equating coefficients of $\exp(-i\psi)$ obviously leads to an equivalent set of equations. System (3.5) may be formally treated as a system of linear algebraic equations relative to η_{jl}. By virtue of the choice of the generating solution

$$\Delta = |(\omega_j^2 - \omega_*^2 + ig_j\omega_*)\,\delta_{jk} + 0_* a_{jk} + \mu_* b_{jk}| = 0 \tag{3.6}$$

In order for system (3.4) to admit of a solution it is necessary and sufficient that

$$\sum_{j=1}^{n} (-2i\omega_* \dot{a}\xi_j + 2\omega_* \dot{\varphi} a\xi_j - g_j \dot{a}\xi_j - ig_j\dot{\varphi} a\xi_j + \Psi_j^{(1)})\chi_j^* = 0 \tag{3.7}$$

Here χ_j is a solution to a system conjugate with respect to system (2.6). Let us denote

$$\frac{\sum_{j=1}^{n} \Psi_j^{(1)}\chi_j^*}{\sum_{j=1}^{n} \xi_j\chi_j^*} = G + iH, \qquad \frac{\sum_{j=1}^{n} g_j\xi_j\chi_j^*}{\sum_{j=1}^{n} \xi_j\chi_j^*} = g$$

where G and H are real functions of a. Condition (3.7) assumes the form

$$-(2i\omega_* + g)\dot{a} + [(2\omega_* - ig)a\dot{\varphi} = G + iH \tag{3.8}$$

By separating the real and imaginary parts, we obtain from here

$$\dot{a} = -\frac{2\omega_* H(a) + gG(a)}{4\omega_*^2 + g^2}, \qquad \dot{\psi} = \omega_* + \frac{2\omega_* G(a) - gH(a)}{a(4\omega_*^2 + g^2)} \tag{3.9}$$

For small g we arrive at the ordinary equations of the Krylov-Bogolyubov method

$$\dot{a} = -\frac{H(a)}{2\omega_*}, \qquad \dot{\psi} = \omega_* + \frac{G(a) - gH(a)}{2a\omega_*} \tag{3.10}$$

We obtain the steady solutions by setting $\dot{a} = 0$, $\dot{\varphi}$ = const. The equations for this case obviously coincide with the equations obtained

by the small parameter method [2].

Instead of using the conjugate system concept, it is possible to proceed in the following manner. Let us consider the determinant of (3.6) denoted by Δ. The generating system admits of a unique periodic solution with a simple exponent equal to $i\omega_*$; therefore the rank of the determinant is obviously n-1. In order for system (3.4) to be solvable with respect to $a\eta_{j1}$, it is necessary and sufficient that the determinant Δ_0, which is obtained from Δ by replacing one of the columns by the column of the right side of

$$V_j = -(2i\omega_* \dot{a} - 2\omega_* \dot{\varphi} a + g_j \dot{a} + i g_j \dot{\varphi} a)\xi_j + \Psi_j^{(1)} \tag{3.11}$$

be equal to 0. After separating the real and imaginary parts we obtain the equations

$$\begin{aligned} \operatorname{Re} \Delta_0 &= \Delta_1 (\dot{a}, \dot{\varphi}, a, \varphi, \mu, \theta) = 0 \\ \operatorname{Im} \Delta_0 &= \Delta_2 (\dot{a}, \dot{\varphi}, a, \varphi, \mu, \theta) = 0 \end{aligned} \tag{3.12}$$

equivalent to Eqs. (3.9).

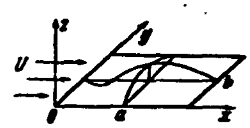

Fig. 6.

4. Example. Flat Unheated Panel. Let us assume that a flat rectangular plate with sides a and b and constant thickness h supported along the perimeter is flowed past by a supersonic flow in the direction of one of the sides (Fig. 6). We shall proceed from the nonlinear Karman equations, neglecting, as is usually done, the tangential components of the inertial forces and considering the perimeter of the panel indeformable. The initial stresses in the

middle surface will be assumed equal to zero. The excess aerodynamic pressure will be determined from the linearized quasi-steady formula of "piston" theory. The deflection w(x, y, t) will be approximated with the aid of two modes of oscillations

$$w(x, y, t) = \left[\zeta_1(t)\sin\frac{\pi x}{a} + \zeta_2(t)\sin\frac{2\pi x}{a}\right] h \sin\frac{\pi y}{b}$$

Under the assumptions that have been made, system (1.1) has the form

$$\ddot{\zeta}_1 + g_1\dot{\zeta}_1 + \omega_1^2\zeta_1 - \frac{2}{3}\mu\zeta_2 = \Phi_1(\zeta_1, \zeta_2), \qquad \ddot{\zeta}_2 + g_2\dot{\zeta}_2 + \omega_2^2\zeta_2 + \frac{2}{3}\mu\zeta_1 = \Phi_2(\zeta_1, \zeta_2) \tag{4.1}$$

Here

$$\omega_1 = 1 + m^2, \qquad \omega_2 = 4 + m^2 \qquad \left(m = \frac{a}{b}\right) \tag{4.2}$$

$$\Phi_1 = -\zeta_1(c_{11}\zeta_1^2 + c_{12}\zeta_2^2), \qquad \Phi_2 = -\zeta_2(c_{21}\zeta_1^2 + c_{22}\zeta_2^2)$$

$$c_{11} = \left[1 + m^4 + \frac{2(1 + 2\nu m^2 + m^4)}{1 - \nu^2}\right]S, \qquad c_{22} = \left[16 + m^4 + \frac{2(16 + 8\nu m^2 + m^4)}{1 - \nu^2}\right]S \tag{4.3}$$

$$c_{12} = c_{21} = \left[4(1 + m^4) + \frac{81m^4}{(1 + 4m^2)^2} + \frac{m^4}{(9 + 4m^2)^2} + \frac{2(4 + 5\nu m^2 + m^4)}{1 - \nu^2}\right]S$$

$$S = \frac{3}{4}(1 - \nu^2) \tag{4.4}$$

For the dimensionless time t we shall take the parameter

$$t = \frac{\pi^2}{a^2}\left(\frac{D}{\rho_0 h}\right)^{1/2} t_1$$

Here t_1 is natural time; D is the cylindrical rigidity; and ρ_0, the density of the panel material. In subsequent calculations we shall assume $g_1 = g_2 = g$.

It is not difficult to deduce the following formulas for the critical parameter μ_* and the frequency of the oscillations at the boundary of the flutter region ω_*:

$$\mu_* = \frac{3}{4}\left[(\omega_2^2 - \omega_1^2)^2 + 2g^2(\omega_1^2 + \omega_2^2)\right]^{1/2}, \qquad \omega_* = \left(\frac{\omega_1^2 + \omega_2^2}{2}\right)^{1/2} \tag{4.5}$$

and also expressions for the constants ξ_1 and ξ_2. Since these constants are related by the formula

$$\frac{2}{3}\mu_*\xi_1 + (\omega_2^2 - \omega_*^2 + ig\omega_*)\xi_2 = 0$$

we may take for them the expressions

$$\xi_1 = \frac{1}{2}, \qquad \xi_2 = -\frac{1}{2} e^{-i\gamma}$$

Here γ is the phase shift between ξ_1 and $-\xi_2$

$$\tan\gamma = \frac{(\omega_1^2 + \omega_2^2)^{1/2} 2^{1/2} g}{\omega_2^2 - \omega_1^2} \tag{4.6}$$

For such a choice of the parameters ξ_1 and ξ_2 an amplitude a = 1 corresponds to a flexure along one halfwave in a direction parallel to the flow with a maximum deflection equal to the thickness of the plate h. The amplitude of the other component in this case will also be equal to h (Fig. 7).

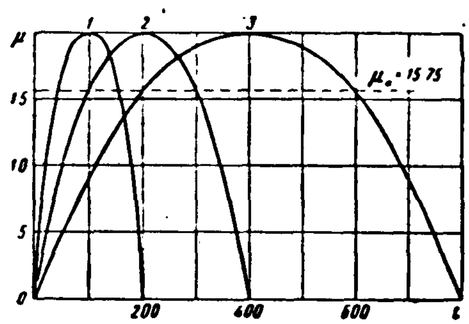

Fig. 7.

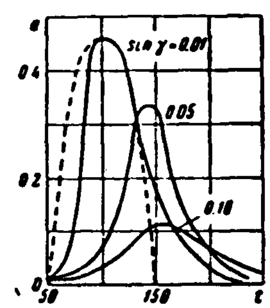

Fig. 8.

Proceeding to construct Eq. (3.9), we shall first write out the determinant in (3.6). In the notations of (4.6) it has the form

$$\Delta = \begin{vmatrix} -e^{-i\gamma} & -1 \\ 1 & e^{i\gamma} \end{vmatrix} = 0 \tag{4.7}$$

Consequently, the condition $\Delta_0 = 0$ obtained from (4.7) by replacing one of the columns by a column made up of the functions (3.10) will be

$$V_1 e^{i\gamma} + V_2 = 0 \tag{4.8}$$

Let us calculate the functions V_1 and V_2. By noting that, when Eqs. (3.4) and (3.5) are taken into account,

$$\begin{aligned}
\zeta_1^3 &= \frac{3}{8} a^3 (e^{i\psi} + e^{-i\psi}) + \ldots \\
\zeta_2^3 &= -\frac{3}{8} a^3 [e^{i(\psi-\gamma)} + e^{-i(\psi-\gamma)}] + \ldots \\
\zeta_1^2\zeta_2 &= -\frac{1}{8} a^3 [(2e^{-i\gamma} + e^{i\gamma}) e^{i\psi} + (2e^{i\gamma} + e^{-i\gamma}) e^{-i\psi}] + \ldots \\
\zeta_1\zeta_2^2 &= \frac{1}{8} a^3 [(2 + e^{-2i\gamma}) e^{i\psi} + (2 + e^{2i\gamma}) e^{i\psi}] + \ldots
\end{aligned}$$

(dots denote the terms containing the harmonics), we find from (3.11) and (4.2)

$$\begin{aligned}
2V_1 &= -(2i\omega_* \dot{a} - 2\omega_* \dot{\varphi} a + g\dot{a} + ig\varphi a) - \frac{2(\mu-\mu_*)a}{3} e^{-i\gamma} - \frac{3a^3}{4}\left(c_{11} + \frac{2 + e^{-2i\gamma}}{3} c_{12}\right) \\
2V_2 &= (2i\omega_* \dot{a} - 2\omega_* \dot{\varphi} a + g\dot{a} + ig\varphi a) e^{-i\gamma} - \frac{2(\mu-\mu_*)a}{3} + \frac{3a^3}{4}\left(\frac{2e^{-i\gamma} + e^{i\gamma}}{3} c_{21} + c_{22} e^{-i\gamma}\right)
\end{aligned}$$

Substitution into Eq. (4.8) yields

$$\begin{aligned}
-(2i\omega_* + g)\dot{a} + (2\omega_* - ig) a\dot{\varphi} = \frac{1}{2i\sin\gamma}\Big[\frac{4}{3}(\mu-\mu_*)a - \\
-\frac{3}{4} a^3 (c_{21} + c_{22} - c_{11} - c_{12})\cos\gamma + \frac{3}{4} a^3 i \left(c_{11} + c_{22} + \frac{c_{12} + c_{21}}{3}\right)\sin\gamma\Big]
\end{aligned}$$

Combining this equation with (3.8), we find that

$$\begin{aligned}
G(a) &= \frac{3}{8} a^3 \left(c_{11} + c_{22} + \frac{c_{12} + c_{21}}{3}\right) \\
H(a) &= -\frac{1}{2\sin\gamma}\left[\frac{4}{3}(\mu-\mu_*)a - \frac{3}{4} a^3 (c_{21} + c_{22} - c_{11} - c_{12})\cos\gamma\right]
\end{aligned} \tag{4.9}$$

To obtain the steady-state amplitude $a = a_0 = \text{const}$ we have the equation $2\omega H(a_0) + gG(a_0) = 0$. Hence we arrive at the formula

$$a_0 = \frac{4}{3}\left(\frac{\mu-\mu_*}{kc_0}\right)^{1/2} \tag{4.10}$$

where $c_0 = c_{21} + c_{22} - c_{11} - c_{12}$, and the coefficient $\underline{k}$ is defined by the formula

$$k = \left(1 + \frac{g\tan\gamma}{2\omega_*}\frac{c_{11} + c_{22} + (c_{12} + c_{11})/3}{c_{21} + c_{22} - c_{11} - c_{12}}\right)\cos\gamma \tag{4.11}$$

As is apparent from (4.6), $\tan\gamma > 0$, $\tan\gamma$ being about equal to $\underline{g}$. If

$$g^2 \ll \omega_2^2 - \omega_1^2 \tag{4.12}$$

we may assume that k ≈ 1, and Eq. (4.10) converts to the well known formula encountered in the literature [1, 2, 6].

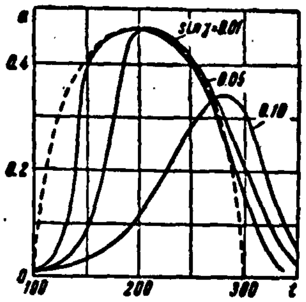

Fig. 9.

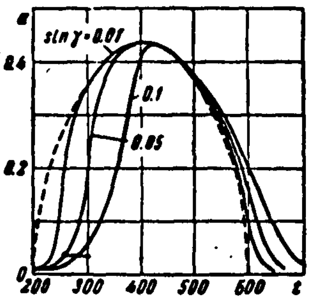

Fig. 10.

Assume condition (4.12) is fulfilled. The first equation in (3.9) takes the form

$$\dot{a} \approx \frac{3c_0 a}{16\omega_* \sin\gamma}(a_0^2 - a^2) \tag{4.13}$$

Shown in Fig. 7 are the three ideal programs for the variation of the parameter μ in time. The results of numerical integration of Eq. (4.13) with the initial condition $a(t_*) = 0.01$ are given in Fig. 8-10 (t_* is the moment of time corresponding to the first intersection of the flutter line). It is assumed that the plate is a square (m = 1), while the Poisson constant $\nu = 0.3$. Under these assumptions

$$-\mu_* \approx 15.75, \quad \omega_* \approx 3.808, \quad c_0 = 35.438.$$

The curves of $a = a_0(t)$, which correspond to the instantaneous establishment of steady solution are plotted by the broken lines. Calculations carried out for the cases sin γ = 0.01, 0.05, and 0.10 demonstrate the strong effect of damping on the amplitude of the non-steady flutter.

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